Phase instabilities generated by parametric modulation in reaction diffusion systems

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Abstract

Effect of external periodic force on an oscillatory order in a reaction diffusion system (Gierer Meinhardt model) has been investigated. The 2:1 resonance situation is found susceptible for the generation of a band of phase instabilities. These phase instabilities, captured on multiple time scales, produces a mismatch between the oscillation frequency of reacting species.

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The effect of parametric periodic forcing on oscillatory reaction diffusion systems are being studied with renewed interest to see frequency entrainment and resulting multiphase, steady as well as traveling, orders separated by phase fronts [1, 2, 3, 4]. Existence of multiphase oscillations are theoretically accounted for by showing the stability of phase separated oscillatory orders in complex Ginzburg Landau equation or in some reaction diffusion models. The mechanism that can possibly cause a slow drift in overall phases of oscillation under periodic forcing, and thus produce stable phase-separated regions, is an important subject for investigation. In view of that, we are going to investigate the effect of periodic forcing in time, on an oscillatory system, on multiple time scales. In almost all reactions diffusion systems, one of the reacting species is dependent on the other for its production and thus does not need be externally supplied. This situation causes

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a constant phase difference to appear between the homogeneous oscillatory reactants of the system. Here we focus on the possible slow variation of that phase difference as a consequence of varied response to the applied force by the reacting species. The reaction diffusion system we have worked on, is the Gierer-Meinhardt model. The results obtained are the generation of a band of phase instabilities caused by parametric resonance at 2:1 sub-harmonic response to the applied frequency. The instabilities developed at slow time scales and produce a mismatch in frequencies of oscillation of the activator and inhibitor species. The result is interesting because in such a situation when activator and inhibitors are oscillating with different frequencies, many different things can follow. An incommensurency in the activator and inhibitor frequencies of oscillation can possibly generate weak phase chaos. In other case, if one of the two species tries to lock itself in phase with the other, a slow drift in the overall phase of oscillation can presumably be an outcome. In order to get rid of such frequency mismatch at various orders, a finer adjustment of the parameter is needed. Such an adjustment also causes a nontrivial shift in phase boundary obtained from the linear stability analysis of the system.

The Gierer-Meinhardt model that we have taken up is [5, 6]

$$\begin{aligned} \frac{\partial A}{\partial t} &= D\bigtriangledown^2 A + \frac{A^2}{B} - A + \sigma \\ \frac{\partial B}{\partial t} &= \bigtriangledown^2 B + \mu(A^2 - B) \end{aligned} \tag{1}$$

D is the diffusivity of the activator A and is always less than unity to satisfy the Turing condition. The σ is the basic production rate of activator where μ can be interpreted as the production constant and at the same time removal rate of the inhibitor B. The linear stability analysis of this model shows that the homogeneous stationary basic fixed point $B = A^2 = (1+\sigma)^2$ which becomes unstable [5] to a time independent spatially inhomogeneous state when

$$\mu D \leq \left(\sqrt{\frac{2}{1+\sigma}} - 1\right)^2 \tag{2}$$

and to a time periodic spatially homogeneous state if

$$\mu < \frac{1-\sigma}{1+\sigma} \tag{3}$$

For the time periodic state the oscillation frequency is [7]

$$\omega_0 = \sqrt{\frac{1-\sigma}{1+\sigma}} \tag{4}$$

Figure 1. shows the phase boundaries in the removal rate μ vs. diffusivity D space. On the left of the broken curve and above the continuous horizontal line the Turing state is stable. In what follows, we will concentrate on the region below the continuous horizontal line in Fig.1 where a Hopf state is available.

Here we are going to consider the problem which is often treated in hydrodynamic instabilities - the one where the control parameter is given a sinusoidal temporal variation [8, 9, 10, 11]. To do so we first linearize Eq.(1) around the fixed point $B = A^2 = (1+\sigma)^2$ and near the boundary $\mu = \frac{1-\sigma}{1+\sigma}$. Thus we have

$$L_0 \begin{pmatrix} \delta A \\ \delta B \end{pmatrix} = 0 \tag{5}$$

where the operator L_0 reads

$$L_0 = \begin{pmatrix} \frac{\partial}{\partial t} - \frac{1-\sigma}{1+\sigma} & \frac{1}{(1+\sigma)^2} \\ -2\mu(1+\sigma) & \frac{\partial}{\partial t} + \mu \end{pmatrix} \tag{6}$$

We note that the state obtained for $\mu < \frac{1-\sigma}{1+\sigma}$ is spatially homogeneous but temporally oscillating. We now endow μ with a small amplitude time modulation

$$\mu = \mu_0 \left(1 + \epsilon Cos(\omega t)\right) \tag{7}$$

and ask the question: what is the critical value of μ_0 for which the spatially homogeneous oscillatory state is formed.

The linearized equation now reads

$$L_0 \begin{pmatrix} \delta A \\ \delta B \end{pmatrix} = \epsilon Cos(\omega t) \begin{pmatrix} 0 & 0 \\ 2\mu_0(1+\sigma) & -\mu_0 \end{pmatrix} \begin{pmatrix} \delta A \\ \delta B \end{pmatrix} \tag{8}$$

The critical value of μ_0 can be expanded as

$$\mu_0 = \mu_{00} + \epsilon \mu_{01} + \epsilon^2 \mu_{02} + ... \tag{9}$$

where $\mu_{00} = \frac{1-\sigma}{1+\sigma}$ and in L_0 it is implied that $\mu = \mu_{00}$. In $O(1)$, the eigenvector for the homogeneous oscillatory state below the afore said Hopf-bifurcation boundary is

$$\begin{pmatrix} \delta A_0 \\ \delta B_0 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2\mu_{00}(1+\sigma)}{\mu_{00}+i\omega_0} \end{pmatrix} e^{i\omega_0 t} + c.c. \tag{10}$$

Thus we see that δA_0 and δB_0 has a constant phase difference since the production of B depends on A. At this point we consider this Phase difference $\phi(\mu)$ has an additive part which varies on a slower time scale $\epsilon\tau$. So the structure of $\phi(\mu)$ is taken as

$$\phi = \phi_c + \delta\phi(\tau) \tag{11}$$

where ϕ_c is the critical value of $\phi(\mu)$ and can be easily obtained from Eq.(10). The $\delta\phi$ is expanded in powers of ϵ as

$$\delta\phi = \delta\phi_0 + \epsilon\delta\phi_1 + \epsilon^2\delta\phi_2 + ... \tag{12}$$

Let us expand δA and δB as

$$\begin{aligned} \delta A &= \delta a_0 + \epsilon\delta a_1 + \epsilon^2\delta a_2 + ... \\ \delta B &= \delta b_0 + \epsilon\delta b_1 + \epsilon^2\delta b_2 + ... \end{aligned} \tag{13}$$

and introduce the multiple time scales as

$$t = t_0 + \epsilon\tau \tag{14}$$

where

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial \tau} \tag{15}$$

We now specialize to the case of parametric resonance i.e. $\omega = 2\omega_0$. The $O(\epsilon)$ equation is

$$L_0 \begin{pmatrix} \delta a_1 \\ \delta b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \delta b_+ \frac{\partial}{\partial\tau}(\delta\phi_0) + \mu_{01}[2(1+\sigma)\delta a_+ - b_+] + \frac{\mu_{00}}{2}[2(1+\sigma)\delta a_- - \delta b_-] \end{pmatrix} \tag{16}$$

On the right hand side of above equation, only the secular terms have been considered. The $\pm$ sign indicates $\pm i\omega$ in the appropriate expressions of amplitudes δa_0 and δb_0 of the $O(0)$ solution. Now, the condition of occurrence of $O(\epsilon)$ solution i.e. the vanishing of secular terms lead to

$$\frac{\partial}{\partial\tau}(\delta\phi_0) = -\frac{\mu_{01}[2(1+\sigma)\delta a_+ - b_+] + \frac{\mu_{00}}{2}[2(1+\sigma)\delta a_- - \delta b_-]}{\delta b_+} \tag{17}$$

Coming back to the original time scale and simplifying

$$\frac{\partial}{\partial t}(\delta\phi_0) = -\frac{i\epsilon\omega_0}{\mu_{00}}\left(\mu_{01} - \frac{\mu_{00}(\mu_{00}-1)}{2(\mu_{00}+1)}\right) - \frac{\epsilon\omega_0^2\mu_{00}}{\mu_{00}^2+\omega_0^2} \tag{18}$$

The condition that there will occur $O(\epsilon)$ solution is the occurrence of a mismatch in frequencies of oscillation of activator and inhibitor by the amount $\frac{\omega_0}{\mu_0 0}\left(\mu_1 - \frac{\mu_0(\mu_0-1)}{2(\mu_0+1)}\right)$. One has to pull the Hopf boundary up by the amount

$$\mu_{01} = \frac{\mu_{00}(\mu_{00}-1)}{2(\mu_{00}+1)} \tag{19}$$

to do away with such frequency mismatch between activator and inhibitor or in other words to get rid of weak disturbances.

The solution of $O(\epsilon)$ equation for $(\omega = 2\omega_0)$ will now definitely include $e^{\pm i3\omega_0}$ term. This term will make secular terms appear in the next higher order. The $O(\epsilon)$ solution in $e^{\pm i3\omega_0}$ will come as

$$\begin{pmatrix} \delta\bar{a}_1 \\ \delta\bar{b}_1 \end{pmatrix} = \frac{\mu_{00}}{2\Delta_3\omega_0}[2(1+\sigma)\delta a_+ - \delta b_+] \begin{pmatrix} \frac{-1}{(1+\sigma)^2} \\ i3\omega_0 - \mu_{00} \end{pmatrix} e^{i3\omega_0 t} + c.c. \tag{20}$$

The $\Delta_{3\omega_0}$ being the determinant of L_0 at frequency $3\omega_0$. Let us look at the $O(\epsilon^2)$ equation with only the secular part (i.e. the part of frequency ω_0) on the right hand side.

$$L_0 \begin{pmatrix} \delta a_2 \\ \delta b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \delta b_+ \frac{\partial}{\partial\tau}(\delta\phi_1) + \mu_{02}[2(1+\sigma)\delta a_+ - b_+] + \frac{\mu_{00}}{2}[2(1+\sigma)\delta\bar{a}_1 - \delta\bar{b}_1] \end{pmatrix} \tag{21}$$

Again the removal of secular terms results in

$$\frac{\partial\phi_1}{\partial t} = -\epsilon\frac{\omega_0}{\mu_{00}}\left[i\left(\mu_{02} + (\frac{\mu_{00}}{2})^2\frac{1}{\Delta_{3\omega_0}}\right) - (\frac{\mu_{00}}{2})^2\frac{3}{\Delta_{3\omega_0}}\right] \tag{22}$$

So the condition for existence of $O(\epsilon^2)$ solution is a further mismatch of frequency of oscillation of the reacting species. A fine tuning of μ by

$$\mu_{02} = (\frac{\mu_{00}}{2})^2\frac{1}{\mu_{00}^2 + 9\omega_0^2 - \frac{2(1+\sigma)}{(1+\sigma)^2}} \tag{23}$$

can help getting rid of this frequency mismatch at this order too. In the above expression we have put

$$\Delta_{3\omega_0} = -(\mu_{00}^2 + 9\omega_0^2 - \frac{2(1+\sigma)}{(1+\sigma)^2}) \tag{24}$$

Thus we see that persistence to retain the frequency ω_0 for the $O(\epsilon^2)$ solution causes a further shift in the Hopf-bifurcation boundary.

The $O(\epsilon^2)$ solution will also have a $e^{\pm i3\omega_0 t}$ part resulting from $\mu_1[2(1+\sigma)\delta\bar{a}_1 - \delta\bar{b}_1]$ which will cause secular term to appear in the immediate higher order. The above mentioned part of the solution in $O(\epsilon^2)$ is of the form

$$\begin{pmatrix} \delta\bar{a}_2 \\ \delta\bar{b}_2 \end{pmatrix} = \frac{\mu_{01}}{2\Delta_3\omega_0}[2(1+\sigma)\delta\bar{a}_1 - \delta\bar{b}_1] \begin{pmatrix} \frac{-1}{(1+\sigma)^2} \\ i3\omega_0 - \mu_{00} \end{pmatrix} e^{i3\omega_0 t} + c.c. \tag{25}$$

This can be written in a simplified form as

$$\begin{pmatrix} \delta\bar{a_2} \\ \delta\bar{b_2} \end{pmatrix} = -\frac{\mu_{01}}{2\Delta_3\omega_0}(i3\omega_0 + 1)\begin{pmatrix} \delta\bar{a_1} \\ \delta\bar{b_1} \end{pmatrix} \tag{26}$$

Again the $O(\epsilon^3)$ solution will contain a $e^{\pm}i3\omega_0 t$ part coming from the term $\mu_1[2(1+\sigma)\delta\bar{a_2} - \delta\bar{b_2}]$ resulting in

$$\begin{pmatrix} \delta\bar{a_3} \\ \delta\bar{b_3} \end{pmatrix} = \left[-\frac{\mu_{01}}{2\Delta_3\omega_0}(i3\omega_0 + 1)\right]^2 \begin{pmatrix} \delta\bar{a_1} \\ \delta\bar{b_1} \end{pmatrix} \tag{27}$$

and so on. In this way at $O(\epsilon)^n$

$$\begin{pmatrix} \delta\bar{a_n} \\ \delta\bar{b_n} \end{pmatrix} = \left[-\frac{\mu_{01}}{2\Delta_3\omega_0}(i3\omega_0 + 1)\right]^{(n-1)} \begin{pmatrix} \delta\bar{a_1} \\ \delta\bar{b_1} \end{pmatrix} \tag{28}$$

Thus we get a nontrivial flow of secular term generators at all orders producing phase instabilities. Such a band of phase instability is generated as a result of sub-harmonic response to the external force. These instabilities would possibly show up in weak phase turbulence under forcing.

In the conclusion we would like to mention that, a multiple scale perturbation analysis of forced Hopf state reveals the presence of instabilities which are responsible for a slow frequency drift between the two reacting species. We would like to focus on the point of instantaneous varied response of activator and inhibitors is the basic cause of generation of such instabilities. This type of differential response to an applied force can always occur when one of the reactants depends on the other for its production and thus allowing for a delay. A persistence of the reacting species for oscillating in unison can cause in a slow overall phase drift of the system. Such oscillatory regions separated by continuous distribution of relative phases has been experimentally observed by Lin et al. in the forced Belousov-Zhabotinsky system at low forcing amplitude [4]. At high enough forcing amplitude they have got well defined π phase separated 2:1 resonant pattern. It is important to note that a gradual increase in the parameter μ, in our analysis, to get rid of spurious instabilities is also the same as enhancing the forcing amplitude. Consideration of large scale spatial phase variations under resonant forcing can easily be shown to result in an inhomogeneous diffusion equation in ϕ with a complex inhomogeneity C originating as a result of external forcing. This type of a situation can result in spatial instabilities in one of the reacting species. Thus other spatio-temporal instabilities at larger scales can also be shown to exist, which influences the situation under forcing.

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Figure caption

Figure 1 shows the phase diagram of the one dimentional Gierer-Meinhardt model on removal rate μ vs. diffusivity D space. The continuous line is a Hopf bifurcation boundary whereas the broken line separates steady Turing state (on the left) from the basic homogeneous steady state.